Simplifying science with similarity

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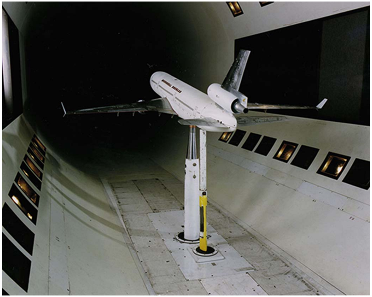
Wind tunnels: Do they "work"? If so, how?

Desired: Drag force F on airplane at specified velocity V and air properties

NASA

McDonnell Douglas MD-11

vs.



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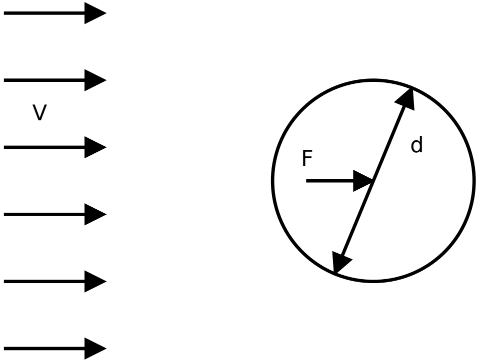
McDonnell Douglas MD-11 small scale model in 12 foot wind tunnel

Is a mathematical predictive model (e.g., the Navier-Stokes equations) required to know this is valid?

Simplified wind tunnel problem

Desired: Drag force F on sphere at specified velocity V, air density ρ, and air viscosity μ



Factorial drag experiment

1 dependent variable:	F
4 independent variables:	V (1 to 100 m/s)
	d (1 to 200 mm)
	ρ (1 to 1000 kg/m^3)
	μ ($9 \cdot 10^{-4}$ to 1.2 Pa $\cdot$ s)

5 in each variable? Only 1 repetition? $\longrightarrow 5 \cdot 5 \cdot 5 \cdot 5 = 625$ runs.

This experiment is not particularly detailed in each variable, and it's still a lot of work. Would you want to do it?

How experiments in wind tunnels are typically done

2 *dimensionless* variables used:

$$\mathrm{Re} = \frac{\rho V d}{\mu} \quad \text{and} \quad C_\mathrm{d} = \frac{F}{\frac{1}{2}\rho V^2 \frac{\pi}{4} d^2}$$

Range of Re: $8 \cdot 10^{-3}$ to $2 \cdot 10^{8}$

5 in Re? Only 1 repetition? $\longrightarrow$ 5 runs. Compare against 625 before.

Is this equivalent to the set up on the previous slide? It potentially involves a lot less work.

Dimensional homogeneity

Let y be a function of n variables:

$$y = f(x_1, x_2, \cdots, x_n) \ .$$

This equation is dimensionally homogeneous if and only if

$$\overline{y} = f(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)$$

where the bar indicates a change of units of measurements.

In other words: The form of the model does not depend on the units of measurement.

Bad correlations

Way to annoy Ben: Make correlations which are not dimensionally homogeneous. E.g. (and this is even worse!):

$$\begin{aligned} \text{for } u'/\overline{U}_0 = 0.30\%, &\quad L_\mathrm{b} = 66 Q_\mathrm{w}^{0.42} \\ \text{for } u'/\overline{U}_0 = 1\%, &\quad L_\mathrm{b} = 38 Q_\mathrm{w}^{0.369} \\ \text{for } u'/\overline{U}_0 = 5\%, &\quad L_\mathrm{b} = 4.6 Q_\mathrm{w}^{0.20} \end{aligned}$$

Buckingham Π theorem Proof: See Langhaar's book or Wikipedia.

An arbitrary dimensionally homogeneous equation

$$0 = f(x_1, x_2, \cdots, x_n) \ .$$

can be expressed in the dimensionless form

$$0 = F(\pi_1, \pi_2, \cdots, \pi_p)$$

where $p = n - k$, and k is the rank of the dimensional matrix. (Example on next slide.)

Typically, k is equal to the number of dimensions in the problem (e.g., mass, length, and time are 3).

Does this theorem require a mathematical predictive model? No! It works a-priori. **No knowledge of f or F is required.**

Buckingham Π theorem applied to wind tunnel example

Variables: F, V, d, ρ, μ ($n = 5$)

Dimensional matrix:

	F	V	d	ρ	μ
M	1	0	0	1	1
L	1	1	1	-3	-1
T	-2	-1	0	0	-1

Rank: 3 (as expected) $\longrightarrow k = 3$

Number of dimensionless variables needed: $n - k = 2$ (one independent, one dependent)

Buckingham Π theorem applied to wind tunnel example

Theory suggests that the Reynolds number is important.

Scaling arguments suggest that the drag coefficient is important.

$$\mathrm{Re} = \frac{\rho V d}{\mu} \quad \text{and} \quad C_\mathrm{d} = \frac{F}{\frac{1}{2}\rho V^2 \frac{\pi}{4} d^2} \quad \longrightarrow C_\mathrm{d} = f(\mathrm{Re})$$

Scaling in a wind tunnel

Assuming that the 4 independent variables identified are all that matters for drag of a sphere, you can predict the drag on a large sphere from measurements of the drag of a small sphere at the same Reynolds number.

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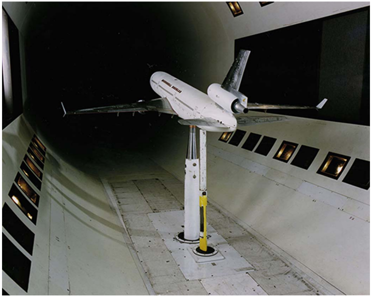
vs.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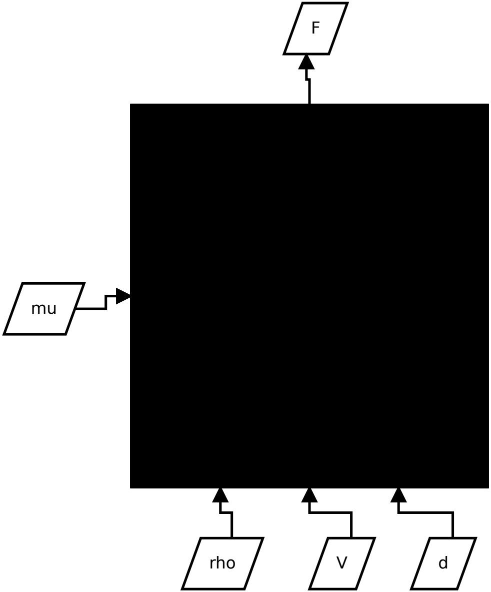
$$\mathrm{Re}_{\mathrm{model}} = \mathrm{Re}_{\mathrm{full}} \quad \text{implies} \quad C_{\mathrm{d\ model}} = C_{\mathrm{d\ full}}$$

Summary of consequences of the Buckingham Π theorem

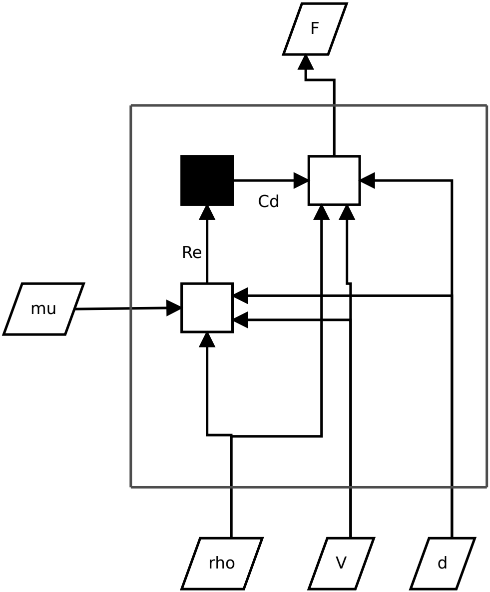
1. Scaling: Each "Π group" suggests how one would develop scale models. "Similarity" is achieved when a model has the same dimensionless numbers as the full scale.
2. Reduction in the number of variables: Typically, the number of dimensionless variables

Both of these consequences can make experiments much easier! Test less variables at the most practical scale.

Buckingham Π: graphical approach



Buckingham Π: graphical approach



When Buckingham Π works well, and when it does not work well

- Great: Small number of known variables $n \leq 3$, many dimensions $\longrightarrow$ Problem entirely specified by dimensional considerations alone!
- Good: Variables known and moderate in number ($n \approx 1$ to 6), many dimensions
  - This is the typical case in fluid mechanics and heat transfer, which may help explain the popularity of the Π theorem there.
- Okay: Variables unknown, many dimensions
- Bad: Variables unknown, data already dimensionless

Problem: Which dimensional variables to pick?

- One (unfair) criticism of this approach is that you don't necessarily know all the relevant variables at the onset. Therefore, Buckingham Π is useless.
- But this is the general case! (E.g., history of wind tunnel operation) Given time and thought, missing variables will typically be found.

Problem: Which dimensional variables to pick?

- However, dimensional analysis actually *can* help: If you are unable to form a dimensionally homogeneous system, then you know that you are either missing variables or need to remove variables.
- Dimensional analysis tells you the consequences of the *hypothesis* that these variables are all that matter.
- Alternative: Look for correlations between everything in extant data, like bioinformatics folks do.

Problem: Which dimensionless variables to pick?

Number of possible independent dimensionless variables:

$$\frac{n!}{k!(n-k)!}$$

Wind tunnel example: 10 (manageable)

Paper I'm writing now: $n = 10$, $k = 3 \longrightarrow 120$ (not impossible, but not ideal)

How do you pick dimensionless variables?

My personal rules for picking dimensionless variables

- Pick dimensionless variables found to be important in theory.
- Non-dimensionalization might make transitions more obvious. Often it is practical to make a “critical” or “transition” occur when a dimensionless variable is 0 or 1.
- Efficiencies are very convenient for quantities involving energy.
- Many dimensionless variables are interpreted as ratios.
- Results in a group that should be approximately constant, e.g., you expect the drag force to the proportional to the frontal area
- Pick dimensionless variables which can be formed from a list of dimensional variables that appears to have the same effect.

Comment: Approaches based on “repeating variables” are systematic, but frequently produce variables which are not found to be practical.

Example: Kolmogorov scale in turbulence

Kolmogorov hypothesized that the smallest length in a turbulent flow (η) depends only on the viscosity ν (L^2/T) and dissipation rate ϵ ($\mathsf{L}^2/\mathsf{T}^3$).

Result:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$

This appears to be roughly correct and is widely used in designing numerical simulations.

Kolmogorov's hypotheses are sometimes invalid, e.g., in combustion.

Example: Principle of corresponding states in thermodynamics

The compressibility factor $Z = \frac{p}{\rho R T}$ is only a function of "reduced" (i.e., dimensionless) pressure p/p_{c}, reduced temperature V/V_{c}, and reduced density/volume/etc. ρ/ρ_{c} for "similar" molecules with some other assumptions.

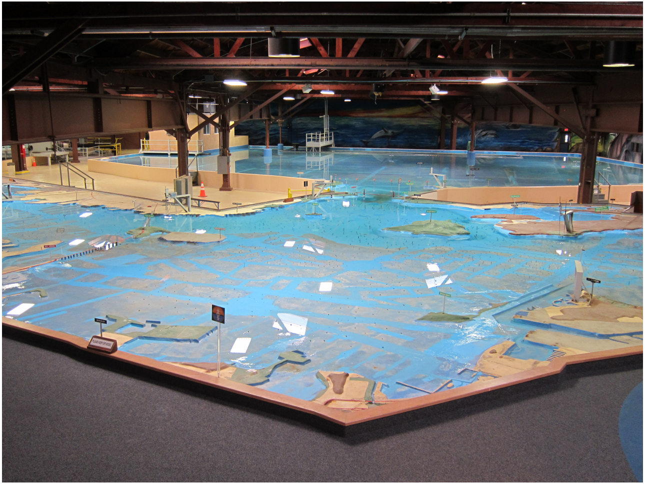
More information: `http://www.sklogwiki.org/SklogWiki/index.php/Law_of_corresponding_states`

Beating the Buckingham Π theorem

The Buckingham Π theorem gives a *sufficient but not necessary* number of variables to ensure similarity.

Theory and experiments may suggest some variables you hypothesized are important don't factor in, or that some variables are less important.

Problem: Broken models



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U.S. Army Corps of Engineers Bay Model

Problem: Temperature in heat transfer

Temperature is frequently treated as a dimension separate from energy. Is this valid?

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Sedov pp. 40-43: In problems where thermal energy is not converted into mechanical energy, and the flow field (velocities) can be found independent of the temperature, temperature can be treated separately from mechanical energy.

It would be interesting to fully generalize this, as it appears the interaction between dimensions is also important. Identical dimensions kept in isolation can be treated separately.

Other similar techniques

dimensional reduction and reducing experimental effort:

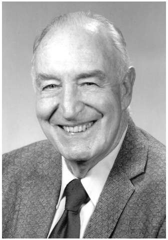
- PCA (and other dimensional reduction techniques, e.g., factor analysis, etc.)
- fractional factorial designs
- theory

getting "something for nothing":

- inequalities
- maximum entropy methods

More on this subject

H. L. Langhaar (1951). *Dimensional Analysis and Theory of Models*. New York: Wiley
Physics-Math-Astronomy Library: QC 39 L35 1951
Library Storage (PCL Stacks): 530.8 L264D c.3
`http://catalog.lib.utexas.edu/record=b2419747~S25`



L. I. Sedov (1959). *Similarity and Dimensional Methods in Mechanics*. New York: Academic Press
PCL - Engineering Collection: QC 39 S413 (checked out)
Physics-Math-Astronomy Library: QC 39 S413 (checked out)
`http://catalog.lib.utexas.edu/record=b1742416~S25`

Questions?

"Surely You're Joking, Mr. Feynman!":

> *So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me.*

Davis, 2011:

> *It seems that the fluid mechanicists have been ahead of the rest of us here, so for those of us working in other areas, perhaps it is about time that we caught up.*
>
> *[...]*
>
> *Dimensional analysis should take its place at the heart of experimental design in engineering applications.*